Multidimensional sum with m-th power of min.

Problem with a solution proposed by Arkady Alt , San Jose , California, USA

For any given natural numbers $m \geq 1, k \geq 2$ prove that

$$\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\ldots\sum_{i_k=1}^{n}\min^m\{i_1,i_2,\ldots,i_k\} = \sum_{i=1}^{m}(-1)^{m-i}\binom{m}{i}\left((n+1)^i - n^i\right)s_{k+m-i}(n),$$

(or $\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\ldots\sum_{i_k=1}^{n}\min^m\{i_1,i_2,\ldots,i_k\} = \sum_{i=1}^{m}\sum_{j=1}^{i}(-1)^{m-i}\binom{m}{i}\binom{i}{j}n^{i-j}s_{k+m-i}(n)$)

where $s_p(n) = \sum_{k=1}^{n}k^p, p \in \mathbb{N}\cup\{0\}$.

Solution.

Let $\sigma_{m,k}(n) := \sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\ldots\sum_{i_k=1}^{n}\min^m\{i_1,i_2,\ldots,i_k\}$.

Since $\min^m\{i_1,i_2,\ldots,i_k\} = \sum_{t=1}^{\min^m\{i_1,i_2,\ldots,i_k\}} 1$ and

$$1 \leq t \leq \min^m\{i_1,i_2,\ldots,i_k\} \Leftrightarrow \begin{cases} 1 \leq t \leq i_1^m \\ 1 \leq t \leq i_2^m \\ \ldots. \\ 1 \leq t \leq i_k^m \end{cases}$$

then $\sigma_{m,k}(n) = \sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\ldots\sum_{i_k=1}^{n}\sum_{t=1}^{\min^m\{i_1,i_2,\ldots,i_k\}} 1 = |D_{ext}|$, where D_{ext} is set of all

$k+1$-tipples $(t,i_1,i_2,\ldots,i_k)$ which satisfy to system of inequalities

$$\begin{cases} 1 \leq i_r \leq n, r = 1,2,\ldots,k \\ 1 \leq t \leq i_r^m, r = 1,2,\ldots,k \end{cases} \Leftrightarrow \begin{cases} 1 \leq t \leq n^m \\ \lceil \sqrt[m]{t}\,\rceil \leq i_r \leq n, r = 1,2,\ldots,k \end{cases}.$$

Hence, $\sigma_{m,k}(n) = \sum_{i=1}^{n^m}\sum_{i_1=\lceil \sqrt[m]{t}\rceil}^{n}\sum_{i_2=\lceil \sqrt[m]{t}\rceil}^{n}\ldots\sum_{i_k=\lceil \sqrt[m]{t}\rceil}^{n} = \sum_{i=1}^{n^m}(n - \lceil \sqrt[m]{t}\,\rceil + 1)^k$.

Since $\{1,2,\ldots,n^m\} = \bigcup_{p=0}^{n-1}\{p^m+1,(p+1)^m\}$ and $\lceil \sqrt[m]{t}\,\rceil = p+1$ for $t \in \{p^m+1,(p+1)^m\}$

then $\sigma_{m,k}(n) = \sum_{p=0}^{n-1}\sum_{t=p^m+1}^{(p+1)^m}(n - \lceil \sqrt[m]{t}\,\rceil + 1)^k = \sum_{p=0}^{n-1}\sum_{t=p^m+1}^{(p+1)^m}(n-p)^k = \sum_{p=0}^{n-1}((p+1)^m - p^m)(n-p)^k$.

In particular, $\sigma_{1,k}(n) = \sum_{p=0}^{n-1}((p+1)-p)(n-p)^k = \sum_{p=0}^{n-1}(n-p)^k = \sum_{p=1}^{n}p^k$,

$\sigma_{2,k}(n) = \sum_{p=0}^{n-1}\left((p+1)^2 - p^2\right)(n-p)^k = \sum_{p=0}^{n-1}(2p+1)(n-p)^k = \sum_{p=1}^{n}(2(n-p)+1)p^k =$

$(2n+1)\sum_{p=1}^{n}p^k - 2\sum_{p=1}^{n}p^{k+1}$.

Note that

$$\sigma_{m,k}(n) = \sum_{p=0}^{n-1}((p+1)^m - p^m)(n-p)^k = \sum_{p=0}^{n-1}(p+1)^m(n+1-(p+1))^k - \sum_{p=0}^{n-1}p^m(n-p)^k =$$

$\sum_{p=1}^{n} p^m(n+1-p)^k - \sum_{p=1}^{n-1} p^m(n-p)^k.$

Let $\delta_{m,k}(n) := \sum_{p=1}^{n-1} p^m(n-p)^k$ then $\sigma_{m,k}(n) = \delta_{m,k}(n+1) - \delta_{m,k}(n)$.

Also note that $\delta_{m,k}(n) = \sum_{p=1}^{n-1} p^m(n-p)^k = \sum_{p=1}^{n-1}(n-(n-p))^m(n-p)^k = \sum_{q=1}^{n-1}(n-q)^m q^k = \delta_{k,m}(n)$

and

$\delta_{m,k+1}(n) + \delta_{m+1,k}(n) = \sum_{p=1}^{n-1} p^m(n-p)^{k+1} + \sum_{p=1}^{n-1} p^{m+1}(n-p)^k = \sum_{p=1}^{n-1} p^m(n-p)^k(n-p+p) = n\delta_{m,k}(n).$

In particular,

$\delta_{1,k}(n) = \sum_{p=1}^{n-1} p(n-p)^k = \sum_{p=1}^{n-1}(n-p)p^k = n\sum_{p=1}^{n-1} p^k - \sum_{p=1}^{n-1} p^{k+1} = ns_k(n-1) - s_{k+1}(n-1)$

(checking $\sigma_{1,k}(n) = \delta_{1,k}(n+1) - \delta_{1,k}(n) = (n+1)s_k(n) - s_{k+1}(n) - ns_k(n-1) + s_{k+1}(n-1) =$
$s_k(n) + n(s_k(n) - s_k(n-1)) - (s_{k+1}(n) - s_{k+1}(n-1)) = s_k(n) + n \cdot n^k - n^{k+1} = s_k(n)$)
and $\delta_{2,k}(n) = n\delta_{1,k}(n) - \delta_{1,k+1}(n) = n(ns_k(n-1) - s_{k+1}(n-1)) - ns_{k+1}(n-1) + s_{k+2}(n-1) =$
$n^2 s_k(n-1) - 2ns_{k+1}(n-1) + s_{k+2}(n-1)$.

Suppose that $\delta_{m,k}(n) = \sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i} n^i s_{k+m-i}(n-1)$ then

$\delta_{m+1,k}(n) = n\delta_{m,k}(n) - \delta_{m,k+1}(n) = \sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i} n^{i+1} s_{k+m-i}(n-1) - \sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i} n^i s_{k+1+m-i}(n$

$\sum_{i=0}^{m}(-1)^{m+1-(i+1)}\binom{m}{i} n^{i+1} s_{k+m+1-(i+1)}(n-1) - \sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i} n^i s_{k+1+m-i}(n-1) =$

$\sum_{i=1}^{m+1}(-1)^{m+1-i}\binom{m}{i-1} n^i s_{k+m+1-i}(n-1) + \sum_{i=1}^{m}(-1)^{m+1-i}\binom{m}{i} n^i s_{k+1+m-i}(n-1) + (-1)^{m+1} s_{k+1+m}(n-1)$

$\sum_{i=1}^{m+1}(-1)^{m+1-i} n^i s_{k+m+1-i}(n-1)\left(\binom{m}{i-1} + \binom{m}{i}\right) + (-1)^{m+1} s_{k+1+m}(n-1) =$

$\sum_{i=1}^{m+1}(-1)^{m+1-i} n^i s_{k+m+1-i}(n-1)\binom{m+1}{i} + (-1)^{m+1} s_{k+m+1}(n-1) = \sum_{i=0}^{m+1}(-1)^{m+1-i}\binom{m+1}{i} n^i s_{k+m+1-i}($

Or, using formal operators, we can obtain the same result much shorter:

Since $\delta_{m+1,k}(n) = n\delta_{m,k}(n) - \delta_{m,k+1}(n)$ and $\delta_{1,k}(n) = ns_k(n-1) - s_{k+1}(n-1)$ then using identical

operator I defined by $I(a_k) = a_k$ and shift operator S defined by $S(a_k) = a_{k+1}$ we obtain
$\delta_{1,k}(n) = (n \cdot I - S)(s_k(n-1)), \delta_{2,k}(n) = (n \cdot I - S)^2(s_k(n-1))$. Since from supposition
$\delta_{m,k}(n) = (nI - S)^m(s_k(n-1))$ follow

$\delta_{m+1,k}(n) = (nI - S)(\delta_{m,k}(n)) = (nI - S)(nI - S)^m(s_k(n-1)) =$
$(nI - S)^{m+1}(s_k(n-1))$. Hence $\sigma_{m,k}(n) = \sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}\left((n+1)^i s_{k+m-i}(n) - n^i s_{k+m-i}(n-1)\right)$.

Since $s_{k+m-i}(n-1) = s_{k+m-i}(n) - n^{k+m-i}$ then

$\sigma_{m,k}(n) = \sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}\left((n+1)^i s_{k+m-i}(n) - n^i(s_{k+m-i}(n) - n^{k+m-i})\right) =$

$\sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}\left(((n+1)^i - n^i)s_{k+m-i}(n) + n^i \cdot n^{k+m-i}\right) = \sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}\left((n+1)^i - n^i\right)s_{k+m-i}$

$$n^{k+m}\sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i} = \sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}\left((n+1)^i - n^i\right)s_{k+m-i}(n) = \sum_{i=1}^{m}(-1)^{m-i}\binom{m}{i}\left((n+1)^i - n\right.$$

Thus,finally

$$\sigma_{m,k}(n) = \sum_{i=1}^{m}(-1)^{m-i}\binom{m}{i}\left((n+1)^i - n^i\right)s_{k+m-i}(n) = \sum_{i=1}^{m}\sum_{j=1}^{i}(-1)^{m-i}\binom{m}{i}\binom{i}{j}n^{i-j}s_{k+m-i}(n).$$

As example we apply obtained formula to determine $\sigma_{3,3}(n)$:

$$\sigma_{3,3}(n) = \sum_{i=1}^{3}(-1)^{3-i}\binom{3}{i}\left((n+1)^i - n^i\right)s_{k+m-i}(n) = \left((n+1)^3 - n^3\right)s_3(n) -$$

$$3\left((n+1)^2 - n^2\right)s_4(n) + 3((n+1)-n)s_5(n) = (3n^2+3n+1)s_3(n) - 3(2n+1)s_4(n) + 3s_5(n).$$

Since $s_4(n) = \dfrac{n(n+1)(2n+1)(3n^2+3n-1)}{30}, s_5(n) = \dfrac{n^2(n+1)^2(2n^2+2n-1)}{12}$ then

$$\sigma_{3,3}(n) = \frac{n^2(n+1)^2(3n^2+3n+1)}{4} - \frac{n(n+1)(2n+1)^2(3n^2+3n-1)}{10} + \frac{n^2(n+1)^2(2n^2+2n-1)}{4}$$

$$\frac{1}{20}n(n+1)(n^2+1)(n^2+2n+2).$$